

# Neutrino mixing and CP violation phases in Zee-Babu model

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We show that the neutrino mass matrix of the Zee-Babu model is able to fit the most recent data on neutrino masses and mixing with large  $\theta_{13}$  and provides the Dirac and Majorana CP violation phases. For the normal hierarchy, the Majorana phases ( $\alpha_{21}, \alpha_{31}$ ) are equal to zero, while for the inverted pattern, one phase ( $\alpha_{31}$ ) takes the value  $2\pi$ . The Dirac phase ( $\delta$ ) is predicted to either 0 or  $\pi$ . The effective mass governing neutrinoless double beta decay and the sum of neutrino masses are consistent with the recent analysis. The model gives some regions of the parameters of neutrino mixing angles in both normal and inverted neutrino mass hierarchy.

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## I. INTRODUCTION

At present, neutrino and Higgs physics are hot topics in current Particle Physics. The neutrino mass and mixing are the first evidence of beyond Standard Model (SM) physics. Despite the Higgs boson has been discovered by the ATLAS and the CMS but in which model it belongs is still open question. For the aforementioned reasons, the search for an extended model coinciding with the current data on neutrino physics is one of our top priorities. In our opinion, the model with the simplest particle content is preferred. By this criterion, the Zee-Babu model [1–3] is very attractive.

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In our previous work [4], we have derived the exact solution for the neutrino mass matrix in the model under consideration and derived some regions of the parameters. In this paper, we focus on the Majorana phases, the effective mass governing neutrinoless double beta decay and the sum of neutrino masses. As we discuss below, the Zee-Babu can fit very well data on recent neutrino experiments.

In this paper, starting from the neutrino mass matrix, we get the exact solution, i.e., the eigenstates and the eigenvalues. As a consequence, the neutrino mixing with large  $\theta_{13}$  is generated. On the other hand, the Dirac CP violation phase is predicted to either 0 or  $\pi$ . In this work, the Majorana phases is included and for the inverted pattern, one phase ( $\alpha_{31}$ ) takes the value  $2\pi$ . The effective mass governing neutrinoless double beta decay and the sum of neutrino mass are consistent with the recent analysis. The model gives some regions of the parameters where neutrino mixing angles and the normal neutrino mass hierarchy obtained consistent with the recent experimental data. With this exact solution, we can fit the most recent data on neutrino mass and mixing.

This paper is organized as follows. In Sec. II, we briefly present the Zee-Babu model and its neutrino mass matrix. Sec. III is devoted for the solution and phenomenology with focus on the Dirac and Majorana violation phases. We summarize our result in the last section - Sec. IV.

## II. ZEE-BABU MODEL AND NEUTRINO MASS MATRIX

With just two  $SU(2)_L$  singlet Higgs fields, a singly charged field  $h^-$  and a doubly charged field  $k^{--}$  and without right-handed neutrinos, the new Yukawa interactions in the Zee-Babu model [2] are

$$\mathcal{L}_Y = f_{ab} \overline{(\psi_{aL})^C} \psi_{bL} h^+ + h'_{ab} \overline{(l_{aR})^C} l_{bR} k^{++} + H.c., \quad (1)$$

where  $\psi_L$  stands for the left-handed lepton doublet,  $l_R$  for the right-handed charged lepton singlet and  $(a, b = e, \mu, \tau)$  being the generation indices, a superscript  $^C$  indicating charge conjugation. Note that  $f_{ab}$  is antisymmetric ( $f_{ab} = -f_{ba}$ ) and  $h'_{ab}$  is symmetric ( $h'_{ab} = h'_{ba}$ ). In terms of the component fields, the interaction Lagrangian is given by

$$\begin{aligned} \mathcal{L}_Y = & 2 \left[ f_{e\mu} (\bar{\nu}_e^c \mu_L - \bar{\nu}_\mu^c e_L) + f_{e\tau} (\bar{\nu}_e^c \tau_L - \bar{\nu}_\tau^c e_L) + f_{\mu\tau} (\bar{\nu}_\mu^c \tau_L - \bar{\nu}_\tau^c \mu_L) \right] h^+ \\ & + [h_{ee} \bar{e}^c e_R + h_{\mu\mu} \bar{\mu}^c \mu_R + h_{\tau\tau} \bar{\tau}^c \tau_R + h_{e\mu} \bar{e}^c \mu_R + h_{e\tau} \bar{e}^c \tau_R + h_{\mu\tau} \bar{\mu}^c \tau_R] k^{++} \\ & + H.c. \end{aligned} \quad (2)$$

where we have used  $h_{aa} = h'_{aa}$ ,  $h_{ab} = 2h'_{ab}$  for  $a \neq b$ . In Eq. (1), the lepton number is conserved, and neutrino mass will be generated due to the Higgs potential given by:

$$V(\phi, h^+, k^{++}) = \mu(h^- h^- k^{++} + h^+ h^+ k^{--}) + \dots \quad (3)$$

Here, the lepton number is violated by two units, hence one expects the Majorana neutrino masses. From Eq.(1), it follows both  $h^-$  and  $k^{--}$  carry lepton number two, so the coefficient  $\mu$  in (3) also carries lepton number two. Therefore it is expected that the Majorana neutrino masses are generated by loop quantum effects. At the two-loop level, the mass matrix for Majorana neutrinos is given by

$$M_{ab} = 8\mu f_{ac} h_{cd}^* m_c m_d I_{cd}(f^+)_{db}, \quad (4)$$

where  $I_{cd}$  has the form [5]

$$I_{cd} = \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{k^2 - m_c^2} \frac{1}{k^2 - M_h^2} \frac{1}{q^2 - m_d^2} \\ \times \frac{1}{q^2 - M_h^2} \frac{1}{(k - q)^2 - M_k^2}. \quad (5)$$

Assuming masses of new Higgses are much larger than lepton ones, we can evaluate  $I_{cd}$  as follows

$$I_{cd} \simeq I = \frac{1}{(16\pi^2)^2} \frac{1}{M^2} \frac{\pi^2}{3} \tilde{I}(r), \quad M \equiv \max(M_k, M_h). \quad (6)$$

Here  $\tilde{I}(r)$  is a function of the ratio of the masses of the charged Higgses  $r \equiv M_k^2/M_h^2$ ,

$$\tilde{I}(r) = \begin{cases} 1 + \frac{3}{\pi^2}(\log^2 r - 1) & \text{for } r \gg 1 \\ 1 & \text{for } r \rightarrow 0, \end{cases} \quad (7)$$

which is close to 1 for a wide range of scalar masses.

The neutrino mass matrix in (4) is symmetric and given by [6]

$$\mathcal{M}_\nu = -I\mu f_{\mu\tau}^2 \times \begin{pmatrix} \epsilon^2 \omega_{\tau\tau} + 2\epsilon\epsilon' \omega_{\mu\tau} + \epsilon'^2 \omega_{\mu\mu} & \epsilon \omega_{\tau\tau} + \epsilon'(\omega_{\mu\tau} - \epsilon \omega_{e\tau} - \epsilon' \omega_{e\mu}) & -\epsilon' \omega_{\mu\mu} - \epsilon(\omega_{\mu\tau} + \epsilon \omega_{e\tau} + \epsilon' \omega_{e\mu}) \\ \star & \omega_{\tau\tau} + \epsilon'^2 \omega_{ee} - 2\epsilon' \omega_{e\tau} & \epsilon \epsilon' \omega_{ee} - \omega_{\mu\tau} - \epsilon \omega_{e\tau} + \epsilon' \omega_{e\mu} \\ \star & \star & \omega_{\mu\mu} + 2\epsilon \omega_{e\mu} + \epsilon'^2 \omega_{ee} \end{pmatrix} \quad (8)$$

where we have redefined parameters:

$$\epsilon \equiv \frac{f_{e\tau}}{f_{\mu\tau}}, \quad \epsilon' \equiv \frac{f_{e\mu}}{f_{\mu\tau}} \quad \omega_{ab} \equiv m_a h_{ab}^* m_b. \quad (9)$$

TABLE I: The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from [8, 9] for normal hierarchy.

Parameter	Best fit	1 $\sigma$ range	2 $\sigma$ range
$\Delta m_{21}^2 (10^{-5} \text{eV}^2)$	7.62	7.43 – 7.81	7.27 – 8.01
$\Delta m_{31}^2 (10^{-3} \text{eV}^2)$	2.55	2.64 – 2.61	2.38 – 2.68
$\sin^2 \theta_{12}$	0.320	0.303 – 0.336	0.29 – 0.35
$\sin^2 \theta_{23}$	0.613	0.573 – 0.635	0.38 – 0.66
$\sin^2 \theta_{13}$	0.0246	0.0218 – 0.0275	0.019 – 0.03

Let us denote [4, 7]

$$\begin{aligned}\omega'_{\tau\tau} &\equiv \omega_{\tau\tau} + \epsilon'^2 \omega_{ee} - 2\epsilon' \omega_{e\tau}, \\ \omega'_{\mu\tau} &\equiv \omega_{\mu\tau} + \epsilon \omega_{e\tau} - \epsilon' \omega_{e\mu} - \epsilon \epsilon' \omega_{ee}, \\ \omega'_{\mu\mu} &\equiv \omega_{\mu\mu} + 2\epsilon \omega_{e\mu} + \epsilon^2 \omega_{ee},\end{aligned}$$

then the neutrino mass matrix can be rewritten in the compact form

$$\mathcal{M}_\nu = -I_\mu f_{\mu\tau}^2 \begin{pmatrix} \epsilon^2 \omega'_{\tau\tau} + 2\epsilon \epsilon' \omega'_{\mu\tau} + \epsilon'^2 \omega'_{\mu\mu} & \epsilon \omega'_{\tau\tau} + \epsilon' \omega'_{\mu\tau} & -\epsilon \omega'_{\mu\tau} - \epsilon' \omega'_{\mu\mu} \\ \star & \omega'_{\tau\tau} & -\omega'_{\mu\tau} \\ \star & \star & \omega'_{\mu\mu} \end{pmatrix}. \quad (10)$$

Next we turn to solution and implication to current neutrino data with a rather large  $\theta_{13}$ .

### III. SOLUTION AND PHENOMENOLOGY

To begin this section, let us present the newest data on neutrino mass and mixing. The best fit values of neutrino mass squared differences and the leptonic mixing angles in [8, 9] have been given to be slightly deviation from Tri-bimaximal mixing form, as shown in Tables I and II with a rather large  $\theta_{13}$ . These large neutrino mixing angles are completely different from the quark mixing ones defined by the Cabibbo- Kobayashi-Maskawa (CKM) matrix [10–12]. This has stimulated works on flavor symmetries and non-Abelian discrete symmetries, which are considered in the models based on the non-Abelian discrete symmetries [13–21] (and the references therein).

The matrix  $\mathcal{M}_\nu$  in (10) has three exact eigenvalues given by

$$\lambda_1 = 0,$$

TABLE II: The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from [8, 9] for inverted hierarchy.

Parameter	Best fit	1 $\sigma$ range	2 $\sigma$ range
$\Delta m_{21}^2 (10^{-5} \text{eV}^2)$	7.62	7.43 – 7.81	7.27 – 8.01
$\Delta m_{13}^2 (10^{-3} \text{eV}^2)$	2.43	2.37 – 2.50	2.29 – 2.58
$\sin^2 \theta_{12}$	0.32	0.303 – 0.336	0.29 – 0.35
$\sin^2 \theta_{23}$	0.60	0.569 – 0.626	0.39 – 0.65
$\sin^2 \theta_{13}$	0.025	0.0223 – 0.0276	0.02 – 0.03

$$\lambda_{2,3} = \frac{1}{2} \left( -kF \pm \sqrt{k^2 \left[ F^2 + 4(1 + \epsilon^2 + \epsilon'^2)(\omega'_{\mu\tau}{}^2 - \omega'_{\mu\mu}\omega'_{\tau\tau}) \right]} \right), \quad (11)$$

where we have denoted

$$k = \mu I f_{\mu\tau}^2, \quad F = (1 + \epsilon'^2)\omega'_{\mu\mu} + 2\epsilon\epsilon'\omega'_{\mu\tau} + (1 + \epsilon^2)\omega'_{\tau\tau}. \quad (12)$$

The massless eigenstate is given by

$$\nu_1 = \frac{1}{\sqrt{f_{e\mu}^2 + f_{e\tau}^2 + f_{\mu\tau}^2}} (f_{\mu\tau}\nu_e - f_{e\tau}\nu_\mu + f_{e\mu}\nu_\tau). \quad (13)$$

Until now values of neutrino masses (or the absolute neutrino masses) as well as the mass ordering of neutrinos are unknown. An upper bound on the absolute value of neutrino mass was found from the analysis of the cosmological data [22]

$$m_i \leq 0.6 \text{ eV}, \quad (14)$$

while arguments from the growth of large-scale structure in the early Universe yield the upper bound on the sum of neutrino mass [23]

$$\sum_{i=1}^3 m_i \leq 0.5 \text{ eV}. \quad (15)$$

The neutrino mass spectrum can be the normal hierarchy ( $|m_1| \simeq |m_2| < |m_3|$ ), the inverted hierarchy ( $|m_3| < |m_1| \simeq |m_2|$ ) or nearly degenerate ( $|m_1| \simeq |m_2| \simeq |m_3|$ ). The mass ordering of neutrino depends on the sign of  $\Delta m_{23}^2$  which is currently unknown. In the case of 3-neutrino mixing, in the model under consideration, the two possible signs of  $\Delta m_{23}^2$  corresponding to two types of neutrino mass spectrum can be provided as follows

1. Normal hierarchy (NH):  $m_1 = 0$ ,  $m_2 = \sqrt{\Delta m_{21}^2}$ ,  $m_3 = \sqrt{\Delta m_{31}^2}$  with  $m_2 \ll m_3$ .
2. Inverted hierarchy (IH):  $m_3 = 0$ ,  $m_1 = \sqrt{\Delta m_{13}^2}$ ,  $m_2 = \sqrt{m_1^2 - \Delta m_{21}^2}$  with  $m_1 \simeq m_2$ .

As we discuss below the neutrino mass matrix in (10) is able to provide both mass schemes

### A. Normal hierarchy ( $\Delta m_{23}^2 > 0$ )

In this case, three neutrino masses are

$$m_1 = \lambda_1 = 0, \quad m_2 = \lambda_2, \quad m_3 = \lambda_3, \quad (16)$$

with  $\lambda_i$  ( $i = 1, 2, 3$ ) is defined in (11), and the corresponding eigenstates put in the neutrino mixing matrix:

$$U_{\nu N} = \begin{pmatrix} \frac{1}{\sqrt{1+\epsilon^2+\epsilon'^2}} & -\frac{A_1}{\sqrt{1+A_1^2+B_1^2}} & \frac{A_2}{\sqrt{1+A_2^2+B_2^2}} \\ -\frac{\epsilon}{\sqrt{1+\epsilon^2+\epsilon'^2}} & -\frac{B_1}{\sqrt{1+A_1^2+B_1^2}} & \frac{B_2}{\sqrt{1+A_2^2+B_2^2}} \\ \frac{\epsilon'}{\sqrt{1+\epsilon^2+\epsilon'^2}} & -\frac{1}{\sqrt{1+A_1^2+B_1^2}} & \frac{1}{\sqrt{1+A_2^2+B_2^2}} \end{pmatrix} \quad (17)$$

where [4]

$$A_{1,2} = \frac{-k \left[ \epsilon(\epsilon'^2 - 1)\omega'_{\mu\mu} + 2\epsilon'(1 + \epsilon^2)\omega'_{\mu\tau} + \epsilon(1 + \epsilon^2)\omega'_{\tau\tau} \right] \pm \epsilon\sqrt{k^2 F'}}{2k \left[ \epsilon\epsilon'\omega'_{\mu\mu} + (1 + \epsilon^2)\omega'_{\mu\tau} \right]}, \quad (18)$$

$$B_{1,2} \equiv \frac{k(1 + \epsilon'^2)\omega'_{\mu\mu} - k(1 + \epsilon^2)\omega'_{\tau\tau} \pm \sqrt{k^2 F'}}{2k \left[ \epsilon\epsilon'\omega'_{\mu\mu} + (1 + \epsilon^2)\omega'_{\mu\tau} \right]}, \quad (19)$$

and

$$F' = F^2 + 4(1 + \epsilon^2 + \epsilon'^2)(\omega_{\mu\tau}'^2 - \omega_{\mu\mu}'\omega_{\tau\tau}'). \quad (20)$$

The eigenstates  $\nu_i$  corresponding to the eigenvalues  $m_i$  ( $i = 1, 2, 3$ ) are found to be

$$\begin{aligned} \nu_1 &= \frac{1}{\sqrt{f_{e\mu}^2 + f_{e\tau}^2 + f_{\mu\tau}^2}}(f_{\mu\tau}\nu_e - f_{e\tau}\nu_\mu + f_{e\mu}\nu_\tau), \\ \nu_2 &= -\frac{A_1}{\sqrt{1 + A_1^2 + B_1^2}}\nu_e - \frac{B_1}{\sqrt{1 + A_1^2 + B_1^2}}\nu_\mu - \frac{1}{\sqrt{1 + A_1^2 + B_1^2}}\nu_\tau, \\ \nu_3 &= \frac{A_2}{\sqrt{1 + A_2^2 + B_2^2}}\nu_e + \frac{B_2}{\sqrt{1 + A_2^2 + B_2^2}}\nu_\mu + \frac{1}{\sqrt{1 + A_2^2 + B_2^2}}\nu_\tau. \end{aligned} \quad (21)$$

From the explicit expressions of  $A_{1,2}$  and  $B_{1,2}$  in (18) and (19), some useful relations are in order [4]

$$\begin{aligned}
A_1 A_2 + B_1 B_2 + 1 &= 0, \\
A_1 - \epsilon B_1 + \epsilon' &= 0, \\
A_2 - \epsilon B_2 + \epsilon' &= 0, \\
(A_1 - A_2)/(B_1 - B_2) &= \epsilon.
\end{aligned} \tag{22}$$

One also has

$$\begin{aligned}
A_1 A_2 &= \frac{(\epsilon'^2 - \epsilon^2)\omega'_{\mu\tau} + \epsilon\epsilon'(\omega'_{\tau\tau} - \omega'_{\mu\mu})}{\epsilon\epsilon'\omega'_{\mu\mu} + (1 + \epsilon^2)\omega'_{\mu\tau}}, \\
B_1 B_2 &= -\frac{(1 + \epsilon'^2)\omega'_{\mu\tau} + \epsilon\epsilon'\omega'_{\tau\tau}}{\epsilon\epsilon'\omega'_{\mu\mu} + (1 + \epsilon^2)\omega'_{\mu\tau}}.
\end{aligned}$$

In the standard Particle Data Group (PDG) parametrization, the neutrino mixing matrix ( $U_{PMNS}$ ) can be parametrized as [24]

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times P, \tag{23}$$

where  $P = \text{diag}\left(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}}\right)$ , and  $c_{ij} = \cos \theta_{ij}$ ,  $s_{ij} = \sin \theta_{ij}$  with  $\theta_{12}$ ,  $\theta_{23}$  and  $\theta_{13}$  being the solar, atmospheric and the reactor angles, respectively, the angles  $\theta_{ij} = [0, \frac{\pi}{2}]$ .  $\delta = [0, 2\pi]$  is the Dirac CP violation phase and  $\alpha_{21}, \alpha_{31}$  are two Majorana CP violation phases. It is to be mentioned that in our previous work [4], the Majorana has been included in a simple form.

By comparing Eqs. (17) and (23), all the parameters in the lepton mixing matrix in (17) can be parameterized in terms of three Euler's angles  $\theta_{ij}$  as follows:

$$A_2 = \frac{c_{12}s_{13}^2\sqrt{1 + \epsilon^2 + \epsilon'^2}}{c_{13}\left(s_{12}s_{23}\sqrt{1 + \epsilon^2 + \epsilon'^2} - \epsilon'\right)}, \quad B_2 = \frac{s_{23}}{c_{23}}, \tag{24}$$

$$A_1 = \frac{A_2 B_1 c_{13}^2 c_{23} s_{12}}{A_2 c_{12} c_{13} c_{23}^2 - s_{12} s_{23} s_{13}^2}, \quad B_1 = \frac{-A_2 \cdot c_{12} c_{13} c_{23}^2 + s_{12} s_{23} s_{13}^2}{c_{23}(s_{12} s_{13}^2 + A_2 \cdot c_{12} c_{13} s_{23})}, \tag{25}$$

$$\begin{aligned}
e^{i\frac{\alpha_{21}}{2}} &= -\frac{1}{s_{12}c_{13}} \frac{A_1}{\sqrt{1 + A_1^2 + B_1^2}}, \\
e^{i\frac{\alpha_{31}}{2}} &= \frac{1}{c_{13}c_{23}} \frac{1}{\sqrt{1 + A_2^2 + B_2^2}}, \\
e^{i\delta} &= \frac{s_{13}}{c_{13}c_{23}A_2},
\end{aligned} \tag{26}$$

and two solutions with  $\epsilon, \epsilon'$ :

$$\epsilon = \frac{s_{12}c_{23} - c_{12}s_{13}s_{23}}{c_{12}c_{13}} \equiv \epsilon_1, \quad \epsilon' = \frac{s_{12}s_{23} + c_{12}s_{13}c_{23}}{c_{12}c_{13}} \equiv \epsilon'_1, \quad (27)$$

or

$$\epsilon = \frac{s_{12}c_{23} + c_{12}s_{13}s_{23}}{c_{12}c_{13}} \equiv \epsilon_2, \quad \epsilon' = \frac{s_{12}s_{23} - c_{12}s_{13}c_{23}}{c_{12}c_{13}} \equiv \epsilon'_2. \quad (28)$$

We are going to consider in detail the first case.

### 1. The solution with $\epsilon_1, \epsilon'_1$

With  $\epsilon = \epsilon_1, \epsilon' = \epsilon'_1$  given in Eq. (27) the parameters in Eqs. from (24) to (26) become:

$$\begin{aligned} A_1 &= -\frac{s_{12}c_{13}}{c_{12}s_{23} - s_{12}s_{13}c_{23}}, \quad A_2 = -\frac{s_{13}}{c_{13}c_{23}}, \\ B_1 &= \frac{s_{12}s_{13}s_{23} + c_{12}c_{23}}{s_{12}s_{13}c_{23} - c_{12}s_{23}}, \quad B_2 = \frac{s_{23}}{c_{23}}, \\ e^{i\frac{\alpha_{21}}{2}} &= 1, \quad e^{i\frac{\alpha_{31}}{2}} = 1, \quad e^{i\delta} = -1. \end{aligned} \quad (29)$$

Taking the data in Refs.[8, 9] given in Tab. I, we obtain

$$A_1 = -0.946222, \quad A_2 = -0.255282, \quad B_1 = -0.986486, \quad B_2 = 1.25856, \quad (30)$$

$$\epsilon = 0.307762, \quad \epsilon' = 0.64262, \quad (31)$$

Eq. (29) implies  $\alpha_{21} = \alpha_{31} = 0, \delta = \pi$ . The neutrino mixing matrix then takes the form:

$$U_{\nu N} = \begin{pmatrix} 0.814415 & 0.558684 & -0.156844 \\ -0.250646 & 0.582457 & 0.773253 \\ 0.523359 & -0.590437 & 0.614394 \end{pmatrix}. \quad (32)$$

Taking the best fit values of the mass squared differences as displayed in Tab.I, we obtain

$$m_2 = 8.72926 \times 10^{-3} \text{ eV}, \quad m_3 = 5.04975 \times 10^{-2} \text{ eV}, \quad (33)$$

and the effective masses  $\langle m_{ee} \rangle, m_\beta$  governing neutrinoless double beta decay [25–29] as well as the sum of the neutrino masses are given by:

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^3 U_{ei}^2 m_i \right| = 0.00396688 \text{ eV}, \quad (34)$$

$$m_\beta = \sum_{i=1}^3 |U_{ei}|^2 m_i^2 = 0.00930129 \text{ eV}, \quad (35)$$

$$m_1 + m_2 + m_3 = 0.0592268 \text{ eV}. \quad (36)$$



Now, substituting  $\epsilon, \epsilon'$  from (31) into (11) and (16) yields:

$$m_1 = 0, \quad m_{2,3} = -\frac{k}{2} \left( 1.41296\omega'_{\mu\mu} + 0.395548\omega'_{\mu\tau} + 1.09472\omega'_{\tau\tau} \mp \sqrt{V_1} \right), \quad (37)$$

where

$$\begin{aligned} V_1 = & 1.99646\omega_{\mu\mu}'^2 + 1.11779\omega_{\mu\mu}'\omega_{\mu\tau}' + 6.18717\omega_{\mu\tau}'^2 \\ & - 2.93713\omega_{\mu\mu}'\omega_{\tau\tau}' + 0.866027\omega_{\mu\tau}'\omega_{\tau\tau}' + 1.19841\omega_{\tau\tau}'^2. \end{aligned} \quad (38)$$

Without loss of generality, we assume  $\omega'_{\mu\mu} = \omega'_{\tau\tau} = \omega'$  [4]. Comparing (33) and (37) we have two solutions:

$$\omega'_{\mu\tau} = -0.771825\omega', \quad k = -\frac{0.0268921}{\omega'}, \quad (39)$$

or

$$\omega'_{\mu\tau} = 0.608604\omega', \quad k = -\frac{0.0215495}{\omega'}. \quad (40)$$

Now we consider the second case.

## 2. The solution with $\epsilon_2, \epsilon'_2$

With  $\epsilon = \epsilon_2, \epsilon' = \epsilon'_2$  given in Eq. (28) the parameters in Eqs. from (24) to (26) become:

$$\begin{aligned} A_1 &= -\frac{s_{12}c_{13}}{c_{12}s_{23} + s_{12}s_{13}c_{23}}, \quad A_2 = \frac{s_{13}}{c_{13}c_{23}}, \\ B_1 &= \frac{s_{12}s_{13}s_{23} - c_{12}c_{23}}{s_{12}s_{13}c_{23} + c_{12}s_{23}}, \quad B_2 = \frac{s_{23}}{c_{23}}, \\ e^{i\frac{\alpha_{21}}{2}} &= 1, \quad e^{i\frac{\alpha_{31}}{2}} = 1, \quad e^{i\delta} = 1. \end{aligned} \quad (41)$$

Taking the data in Refs.[8, 9] given in Tab. I, we obtain

$$A_1 = -0.797179, \quad A_2 = 0.255282, \quad B_1 = -0.63286, \quad B_2 = 1.25856, \quad (42)$$

$$\epsilon = 0.556439, \quad \epsilon' = 0.445031. \quad (43)$$

Eq. (57) implies  $\alpha_{21} = \alpha_{31} = \delta = 0$ , i.e, the Dirac and Majorana CP violation phases are equal to zero.

The neutrino mixing matrix then takes the form:

$$U_{\nu N} = \begin{pmatrix} 0.814415 & 0.558684 & 0.156844 \\ -0.453172 & 0.443525 & 0.773253 \\ 0.36244 & -0.700826 & 0.614394 \end{pmatrix}. \quad (44)$$

In this case, the neutrino masses are given in (33), and the effective masses  $\langle m_{ee} \rangle, m_\beta$  governing neutrinoless double beta decay as well as the sum of the neutrino masses are given by (34), (35) and (36).

Substituting  $\epsilon, \epsilon'$  from (31) into (11), (16) yields:

$$m_1 = 0, \quad m_{2,3} = -\frac{k}{2} \left( 1.19805\omega'_{\mu\mu} + 0.495265\omega'_{\mu\tau} + 1.30962\omega'_{\tau\tau} \mp \sqrt{V_2} \right), \quad (45)$$

where

$$\begin{aligned} V_2 = & 1.43533\omega_{\mu\mu}'^2 + 1.18671\omega_{\mu\mu}'\omega_{\mu\tau}' + 6.276\omega_{\mu\tau}'^2 \\ & - 2.89271\omega_{\mu\mu}'\omega_{\tau\tau}' + 1.29722\omega_{\mu\tau}'\omega_{\tau\tau}' + 1.71512\omega_{\tau\tau}'^2. \end{aligned} \quad (46)$$

In similarity, we assume again  $\omega'_{\mu\mu} = \omega'_{\tau\tau} = \omega'$  [4], and by comparing (33) and (45) we have two solutions:

$$\omega'_{\mu\tau} = -0.671538\omega', \quad k = -\frac{0.00518566}{\omega'}, \quad (47)$$

or

$$\omega'_{\mu\tau} = 0.391254\omega', \quad k = -\frac{0.00417526}{\omega'}. \quad (48)$$

To finish this section, we note that in the normal scheme, the Majorana violation phases are equal to the zero.

### B. Inverted hierarchy ( $\Delta m_{23}^2 < 0$ )

In this case, three neutrino masses are

$$m_1 = \lambda_3, \quad m_2 = \lambda_2, \quad m_3 = 0, \quad (49)$$

with  $\lambda_i$  ( $i = 1, 2, 3$ ) is defined in (11), and the corresponding eigenstates put in the neutrino mixing matrix:

$$U_{\nu I} = \begin{pmatrix} \frac{A_2}{\sqrt{1+A_2^2+B_2^2}} & -\frac{A_1}{\sqrt{1+A_1^2+B_1^2}} & \frac{1}{\sqrt{1+\epsilon^2+\epsilon'^2}} \\ \frac{B_2}{\sqrt{1+A_2^2+B_2^2}} & -\frac{B_1}{\sqrt{1+A_1^2+B_1^2}} & -\frac{\epsilon}{\sqrt{1+\epsilon^2+\epsilon'^2}} \\ \frac{1}{\sqrt{1+A_2^2+B_2^2}} & -\frac{1}{\sqrt{1+A_1^2+B_1^2}} & \frac{\epsilon'}{\sqrt{1+\epsilon^2+\epsilon'^2}} \end{pmatrix} \quad (50)$$

where  $A_{1,2}$  and  $B_{1,2}$  are given in (18) and (19), respectively.

The eigenstates  $\nu_i$  corresponding to the eigenvalues  $m_i$  ( $i = 1, 2, 3$ ) are found to be

$$\begin{aligned}\nu_1 &= \frac{A_2}{\sqrt{1+A_2^2+B_2^2}}\nu_e + \frac{B_2}{\sqrt{1+A_2^2+B_2^2}}\nu_\mu + \frac{1}{\sqrt{1+A_2^2+B_2^2}}\nu_\tau, \\ \nu_2 &= -\frac{A_1}{\sqrt{1+A_1^2+B_1^2}}\nu_e - \frac{B_1}{\sqrt{1+A_1^2+B_1^2}}\nu_\mu - \frac{1}{\sqrt{1+A_1^2+B_1^2}}\nu_\tau, \\ \nu_3 &= \frac{1}{\sqrt{f_{e\mu}^2+f_{e\tau}^2+f_{\mu\tau}^2}}(f_{\mu\tau}\nu_e - f_{e\tau}\nu_\mu + f_{e\mu}\nu_\tau).\end{aligned}\quad (51)$$

Similar to the normal case, comparing Eqs. (50) and (23) yields:

$$\begin{aligned}\epsilon &= \frac{c_{13} \left( \frac{B_2}{\sqrt{1+A_2^2+B_2^2}} + s_{12}c_{23} \right)}{c_{12}s_{13}^2}, \quad \epsilon' = -\frac{c_{23}}{s_{23}}\epsilon, \\ e^{i\frac{\alpha_{21}}{2}} &= -\frac{1}{s_{12}c_{13}} \frac{A_1}{\sqrt{1+A_1^2+B_1^2}}, \\ e^{i\frac{\alpha_{31}}{2}} &= -\frac{1}{c_{13}s_{23}} \frac{\epsilon}{\sqrt{1+\epsilon^2+\epsilon'^2}}, \\ e^{i\delta} &= -\frac{s_{13}}{c_{13}s_{23}}\epsilon,\end{aligned}\quad (52)$$

and two solutions with  $A_2, B_2$ :

$$\begin{aligned}A_2 &= \frac{c_{12}c_{13}}{s_{12}s_{23} + c_{12}s_{13}c_{23}} \equiv A_2^+, \\ B_2 &= \frac{c_{12}s_{12}c_{13} + s_{23}c_{23}(c_{12}^2c_{13}^2 - 1)}{s_{12}^2 + c_{23}^2(c_{12}^2c_{13}^2 - 1)} \equiv B_2^+, \end{aligned}\quad (53)$$

or

$$\begin{aligned}A_2 &= \frac{c_{12}s_{12}c_{13}}{s_{12}^2s_{23} - c_{12}s_{13}c_{23}} \equiv A_2^-, \\ B_2 &= \frac{-c_{12}s_{12}c_{13} + s_{23}c_{23}(c_{12}^2c_{13}^2 - 1)}{s_{12}^2 + c_{23}^2(c_{12}^2c_{13}^2 - 1)} \equiv B_2^-.\end{aligned}\quad (54)$$

Let us consider both cases.

### 1. The solution with $A_2^+, B_2^+$

As is easily shown that, in this case, the model is consistent because the five experimental constraints on the mixing angles and squared mass differences of neutrinos can be respectively fitted with all parameters of the model. Indeed, with  $A_2 = A_2^+, B_2 = B_2^+$  given in Eq. (53), taking the data in Refs.[8, 9] given in Tab. II, we obtain

$$A_1 = -0.959444, \quad A_2 = 1.56394, \quad B_1 = -1.01484, \quad B_2 = -0.493193, \quad (55)$$

$$\epsilon = 4.83735, \quad \epsilon' = -3.94968, \quad \frac{\epsilon}{\epsilon'} = -1.22474, \quad (56)$$

$$e^{i\frac{\alpha_{21}}{2}} = 1, \quad e^{i\frac{\alpha_{31}}{2}} = -1, \quad e^{i\delta} = -1. \quad (57)$$

Eq. (57) implies  $\alpha_{21} = 0$ ,  $\alpha_{31} = 2\pi$ ,  $\delta = \pi$ . The neutrino mixing matrix then takes the form:

$$U_{\nu I} = \begin{pmatrix} 0.814248 & 0.55857 & 0.158114 \\ -0.256776 & 0.590818 & -0.764853 \\ 0.52064 & -0.58218 & -0.6245 \end{pmatrix}. \quad (58)$$

The physical neutrino masses are obtained as

$$\begin{aligned} m_1 &= \sqrt{\Delta m_{13}^2} = 4.9295 \times 10^{-2} \text{ eV}, \\ m_2 &= \sqrt{m_1^2 - \Delta m_{21}^2} = 4.8516 \times 10^{-2} \text{ eV}, \quad m_3 = 0, \end{aligned} \quad (59)$$

and the effective masses  $\langle m_{ee} \rangle, m_\beta$  governing neutrinoless double beta decay [25–29] as well as the sum of the neutrino masses are given by:

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^3 U_{ei}^2 m_i \right| = 0.0478196 \text{ eV}, \quad (60)$$

$$m_\beta = \sum_{i=1}^3 |U_{ei}|^2 m_i^2 = 0.0484301 \text{ eV}, \quad (61)$$

$$m_1 + m_2 + m_3 = 0.097811 \text{ eV}. \quad (62)$$

As before, we assume  $\omega'_{\mu\mu} = \omega'_{\tau\tau} = \omega'$  [4]. Substituting  $\epsilon, \epsilon'$  in Eq. (56) into (11) and (49) yields

$$\omega'_{\mu\tau} = 1.07194\omega', \quad k = -\frac{0.0200278}{\omega'}, \quad (63)$$

or

$$\omega'_{\mu\tau} = 1.07399\omega', \quad k = -\frac{0.0197385}{\omega'}. \quad (64)$$

Eq. (57) shows that in this case one Majorana violation phase is nonzero. Our next step is the second case.

## 2. The solution with $A_2^-, B_2^-$

In this case, taking the data in [8, 9] given in Tab. II, we obtain

$$A_1 = -0.80333, \quad A_2 = 2.28904, \quad B_1 = -0.650428, \quad B_2 = -1.2897, \quad (65)$$

$$\epsilon = -4.83735, \quad \epsilon' = 3.94968, \quad \frac{\epsilon}{\epsilon'} = -1.22474, \quad (66)$$

$$e^{i\frac{\alpha_{21}}{2}} = e^{i\frac{\alpha_{31}}{2}} = e^{i\delta} = 1. \quad (67)$$

Eq. (67) implies  $\alpha_{21} = \alpha_{31} = \delta = 0$ . Then, the neutrino mixing matrix is

$$U_{\nu I} = \begin{pmatrix} 0.814248 & 0.55857 & 0.158114 \\ -0.458766 & 0.452254 & 0.764853 \\ 0.355716 & -0.695317 & 0.6245 \end{pmatrix}. \quad (68)$$

Three neutrino masses are given in (62), and the effective masses  $\langle m_{ee} \rangle, m_\beta$  governing neutrinoless double beta decay as well as the sum of the neutrino masses are given by (60), (61) and (62). The relation between  $\omega'_{\mu\tau}, k$  and  $\omega'$  are given in Eqs.(63) and (64). Note that in this case, all Dirac and Majorana violation phases vanish.

#### IV. CONCLUSION

In this paper we have derived the exact eigenvalues and eigenstates of the neutrino mass matrix in the Zee-Babu model in which the most recent data on neutrino masses and mixing with large  $\theta_{13}$  are updated, and with focus on Majorana violation phases. For the normal hierarchy, the Majorana phases ( $\alpha_{21}, \alpha_{31}$ ) are equal to zero, while for the inverted pattern, one phase ( $\alpha_{31}$ ) takes the value  $2\pi$ . The Dirac phase ( $\delta$ ) is predicted to either 0 or  $\pi$ . Taking into account of the effective mass governing neutrinoless double beta decay and the sum of neutrino, we have showed that the model fits well with the recent most experimental data. Therefore we conclude that the Zee-Babu model is fascinating one for neutrino physics.

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